DAY TWENTY ONE

Properties of Triangle, Height and Distances

Learning & Revision for the Day

- Properties Related to Triangle
- Some Important Theorems
- Circles Connected with Triangle
- Angle of Elevation and Depression

Properties Related to Triangle

In any ΔABC,

(i) perimeter, $2s = a + b + c$

(ii) sum of all angles of a triangle is 180°, i.e. $\angle A + \angle B + \angle C = 180°$

(iii) $a + b > c, b + c > a, c + a > b$

(iv) $|a - b| < c, |b - c| < a, |c - a| < b$

(v) $a > 0, b > 0, c > 0$

Relations between the Sides and Angles of Triangle

For a triangle ΔABC with sides a, b, c and opposite angles are respectively A, B and C, then

(i) **Sine Rule** In any ΔABC, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

(ii) **Cosine Rule**

(a) $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$ (b) $\cos B = \dfrac{c^2 + a^2 - b^2}{2ca}$

(c) $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

(iii) **Projection Rule**

(a) $a = b \cos C + c \cos B$ (b) $b = c \cos A + a \cos C$

(c) $c = a \cos B + b \cos A$

(iv) **Napier's Analogy**

(a) $\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2}$ (b) $\tan \frac{C-A}{2} = \frac{c-a}{c+a} \cot \frac{B}{2}$

(c) $\tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2}$

(v) **Half Angle of Triangle**

(a) $\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$ (b) $\sin \frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}$

(c) $\sin \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$ (d) $\cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$

(e) $\cos \frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}$ (f) $\cos \frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$

(g) $\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$ (h) $\tan \frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$

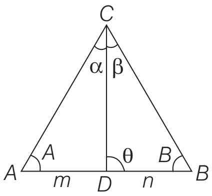
(i) $\tan \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

(vi) Area of a Triangle $\Delta = \frac{1}{2} bc \sin A = \frac{1}{2} ca \sin B = \frac{1}{2} ab \sin C$
$= \sqrt{s(s-a)(s-b)(s-c)}$

Some Important Theorems

1. m-n Theorem (Trigonometric Theorem)

If in a $\Delta ABC, D$ divides AB in the ratio $m : n$, then (shown as in given figure)

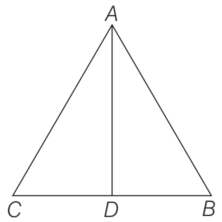


(i) $(m+n) \cot \theta = n \cot A - m \cot B$

(ii) $(m+n) \cot \theta = m \cot \alpha - n \cot \beta$

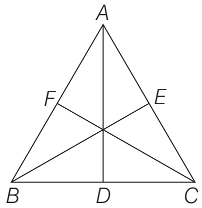
2. Appolonius Theorem

If in ΔABC, AD is median, then



$$AB^2 + AC^2 = 2(AD^2 + BD^2)$$

The length of medians AD, BE and CF of a ΔABC are (shown as in given figure)



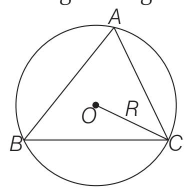
$$AD = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}, \quad BE = \frac{1}{2}\sqrt{2c^2 + 2a^2 - b^2}$$

and $CF = \frac{1}{2}\sqrt{2a^2 + 2b^2 - c^2}$

Circles Connected with Triangle

1. Circumcircle

The circle passing through the vertices of the ΔABC is called the circumcircle. (shown as in given figure)



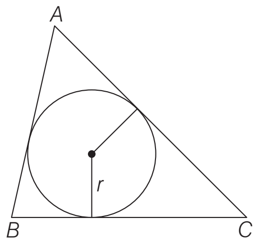
Its radius R is called the circumradius,

and $R = \frac{a}{2 \sin A} = \frac{b}{2 \sin B} = \frac{c}{2 \sin C} = \frac{abc}{4\Delta}$

NOTE
- The mid-point of the hypotenuse of a right angled triangle is equidistant from the three vertices of the triangle.
- The mid-point of the hypotenuse of a right angled triangle is the circumcentre of the triangle.
- Distance of circumcentre from the side AC is $R \cos B$.
- Radius of circumcircle of a n-sided regular polygon with each side a is $R = \frac{a}{2} \text{cosec} \frac{\pi}{n}$.

2. Incircle

The circle touching the three sides of the triangle internally is called the inscribed circle or the incircle of the triangle. Its



radius r is called inradius of the circle.

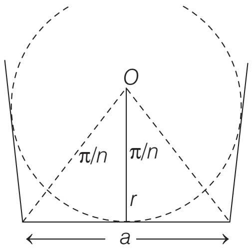
(i) $r = \frac{\Delta}{s}$

(ii) $r = (s-a) \tan \frac{A}{2} = (s-b) \tan \frac{B}{2} = (s-c) \tan \frac{C}{2}$

(iii) $r = 4R \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}$

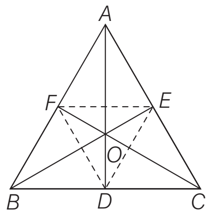
(iv) $r = \dfrac{a \sin\frac{B}{2} \sin\frac{C}{2}}{\cos\frac{A}{2}} = \dfrac{b \sin\frac{C}{2} \sin\frac{A}{2}}{\cos\frac{B}{2}} = \dfrac{c \sin\frac{A}{2} \sin\frac{B}{2}}{\cos\frac{C}{2}}$

NOTE Radius of incircle of a n-sided regular polygon with each side a is $r = \frac{a}{2}\cot\frac{\pi}{n}$. (shown as in given figure)



3. Orthocentre and Pedal Triangle

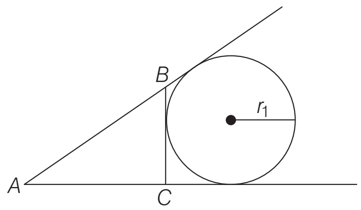
- The point of intersection of perpendiculars drawn from the vertices on the opposite sides of a triangle is called orthocentre. (shown as in given figure)



- The ΔDEF formed by joining the feet of the altitudes is called the pedal triangle.
- Orthocentre of the triangle is the incentre of the pedal triangle.
- Distance of the orthocentre of the triangle from the angular points are $2R\cos A$, $2R\cos B$, $2R\cos C$ and its distances from the sides are $2R\cos B\cos C$, $2R\cos C\cos A$, $2R\cos A\cos B$.

4. Escribed Circle

The circle touching BC and the two sides AB and AC produced of ΔABC, is called the escribed circle opposite to A. Its radius is denoted by r_1. (shown as in given figure)



Similarly, r_2 and r_3 denote the radii of the escribed circles opposite to angles B and C, respectively.

r_1, r_2 and r_3 are called the exradius of ΔABC.

(i) $r_1 = \dfrac{\Delta}{s-a} = s\tan\dfrac{A}{2} = 4R\sin\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2} = \dfrac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}$

(ii) $r_2 = \dfrac{\Delta}{s-b} = s\tan\dfrac{B}{2} = 4R\sin\dfrac{B}{2}\cos\dfrac{C}{2}\cos\dfrac{A}{2} = \dfrac{b\cos\frac{C}{2}\cos\frac{A}{2}}{\cos\frac{B}{2}}$

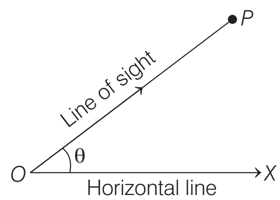
(iii) $r_3 = \dfrac{\Delta}{s-c} = s\tan\dfrac{C}{2} = 4R\sin\dfrac{C}{2}\cos\dfrac{A}{2}\cos\dfrac{B}{2} = \dfrac{c\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{C}{2}}$

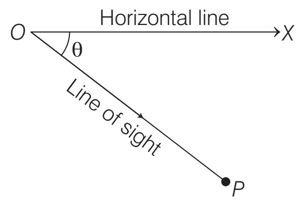
NOTE
- $r_1 + r_2 + r_3 = 4R + r$
- $r_1 r_2 + r_2 r_3 + r_3 r_1 = \dfrac{r_1 r_2 r_3}{r}$
- $(r_1 - r)(r_2 - r)(r_3 - r) = 4Rr^2$
- $\dfrac{1}{r_1} + \dfrac{1}{r_2} + \dfrac{1}{r_3} = \dfrac{1}{r}$

Angle of Elevation and Depression

Let O be the observer's eye and OX be the horizontal line through O. (shown as in following figures)

If a object P is at a higher level than eye, then $\angle POX$ is called the **angle of elevation**.



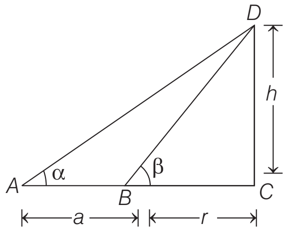


If a object P is at a lower level than eye, then $\angle POX$ is called the **angle of depression**.

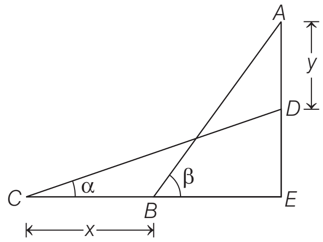
Important Results on Heights and Distances

Results shown by the following figures.

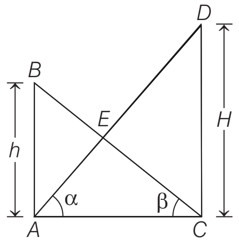
(i) $a = h(\cot\alpha - \cot\beta)$



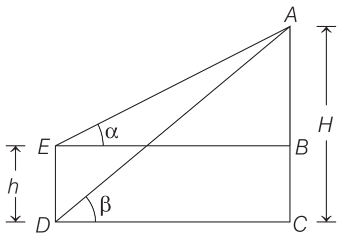
(ii) If $AB = CD$, then $x = y \tan\left(\frac{\alpha+\beta}{2}\right)$



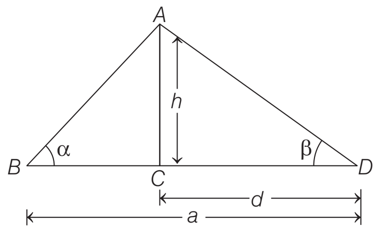
(iv) $H = \frac{h \cot\beta}{\cot\alpha}$



(iii) $h = \frac{H \sin(\beta-\alpha)}{\cos\alpha \sin\beta}$ and $H = \frac{h\cot\alpha}{\cot\alpha - \cot\beta}$



(v) $a = h(\cot\alpha + \cot\beta)$, $h = a\sin\alpha\sin\beta\,\text{cosec}\,(\alpha+\beta)$ and $d = h\cot\beta = a\sin\alpha\cos\beta\,\text{cosec}\,(\alpha+\beta)$



DAY PRACTICE SESSION 1

FOUNDATION QUESTIONS EXERCISE

1 If a, b and c are sides of a triangle, then
(a) $\sqrt{a}+\sqrt{b}>\sqrt{c}$
(b) $\left|\sqrt{a}-\sqrt{b}\right|>\sqrt{c}$ (if c is smallest)
(c) $\sqrt{a}+\sqrt{b}<\sqrt{c}$
(d) None of the above

2 If in a ΔABC, $A = 30°$, $B = 45°$ and $a = 1$, then the values of b and c are respectively
(a) $\sqrt{2}, \frac{\sqrt{3}+1}{\sqrt{2}}$ (b) $\sqrt{2}, \frac{\sqrt{3}-1}{\sqrt{2}}$
(c) $\sqrt{3}, \frac{\sqrt{3}-1}{\sqrt{2}}$ (d) $\sqrt{2}, \frac{\sqrt{3}+2}{\sqrt{2}}$

3 If $A = 75°$, $B = 45°$, then $b + c\sqrt{2}$ is equal to
(a) $2a$ (b) $2a+1$ (c) $3a$ (d) $2a-1$

4 If in a ΔABC, $2b^2 = a^2 + c^2$, then $\frac{\sin 3B}{\sin B}$ is equal to
(a) $\frac{c^2-a^2}{2ca}$ (b) $\frac{c^2-a^2}{ca}$
(c) $\left(\frac{c^2-a^2}{ca}\right)^2$ (d) $\left(\frac{c^2-a^2}{2ca}\right)^2$

5 The sides of a triangle are $\sin\alpha$, $\cos\alpha$ and $\sqrt{1+\sin\alpha\cos\alpha}$ for some $0<\alpha<\frac{\pi}{2}$. Then, the greatest angle of the triangle is
(a) 60° (b) 90° (c) 120° (d) 150°

6 In a ΔABC, $(c+a+b)(a+b-c) = ab$. The measure of $\angle C$ is
(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{6}$
(c) $\frac{2\pi}{3}$ (d) None of these

7 In ΔABC, if $\tan\frac{A}{2}\tan\frac{C}{2} = \frac{1}{2}$, then a, b and c are in
(a) AP (b) GP
(c) HP (d) None of these

8 In a ΔABC, $a:b:c = 4:5:6$. The ratio of the radius of the circumcircle to that of the incircle is
(a) $\frac{16}{9}$ (b) $\frac{16}{7}$ (c) $\frac{11}{7}$ (d) $\frac{7}{16}$

9 In any ΔABC, $4\left(\frac{s}{a}-1\right)\left(\frac{s}{b}-1\right)\left(\frac{s}{c}-1\right)$ is equal to
(a) $\frac{r}{R}$ (b) $\frac{2r}{R}$
(c) $\frac{3r}{R}$ (d) None of these

10 In a ΔABC, R = circumradius and r = inradius. The value of $\frac{a\cos A + b\cos B + c\cos C}{a+b+c}$ is
(a) $\frac{R}{r}$ (b) $\frac{R}{2r}$ (c) $\frac{r}{R}$ (d) $\frac{2r}{R}$

11 Let the orthocentre and centroid of a triangle be $A(-3, 5)$ and $B(3, 3)$, respectively. If C is the circumcentre of this triangle, then the radius of the circle having line segment AC as diameter, is

(a) $\sqrt{10}$ (b) $2\sqrt{10}$ (c) $3\sqrt{\frac{5}{2}}$ (d) $\frac{3\sqrt{5}}{2}$

12 In a triangle $\left(1-\frac{r_1}{r_2}\right)\left(1-\frac{r_1}{r_3}\right)=2$, then the triangle is

(a) right angled (b) equilateral
(c) isosceles (d) None of these

13 There are two stations A and B due North, due South of a tower of height 15 m. The angle of depression of A and B as seen from top of the tower are $\cot^{-1}\left(\frac{12}{5}\right)$, $\sin^{-1}\left(\frac{3}{5}\right)$, then the distance between A and B is

(a) 48 m (b) 56 m (c) 25 m (d) None of these

14 A house of height 100 m subtends a right angle at the window of an opposite house. If the height of the window be 64 m, then the distance between the two houses is

(a) 48 m (b) 36 m (c) 54 m (d) 72 m

15 The angle of elevation of the top of a tower from the top and bottom of a building of height 'a' are 30° and 45°, respectively. If the tower and the building stand at the same level, then the height of the tower is

(a) $\frac{a(3+\sqrt{3})}{2}$ (b) $a(\sqrt{3}+1)$ (c) $a\sqrt{3}$ (d) $a(\sqrt{3}-1)$

16 A person walking along a straight road observes that at two points 1 km apart, the angles of elevation of a pole in front of line are 30° and 75°. The height of the pole is

(a) $250(\sqrt{3}+1)$ m (b) $250(\sqrt{3}-1)$ m
(c) $225(\sqrt{2}-1)$ m (d) $225(\sqrt{2}+1)$ m

17 A man is walking towards a vertical pillar in a straight path, at a uniform speed. At a certain point A on the path, he observes that the angle of elevation of the top of the pillar is 30°. After walking for 10 min from A in the same direction, at a point B, he observes that the angle of elevation of the top of the pillar is 60°. Then, the time taken (in minutes) by him, from B to reach the pillar, is **→ JEE Mains 2016**

(a) 6 (b) 10 (c) 20 (d) 5

18 A bird is sitting on the top of a vertical pole 20 m high and its elevation from a point O on the ground is 45°. It flies off horizontally straight away from the point O. After 1s, the elevation of the bird from O is reduced to 30°.Then, the speed (in m/s)of the bird is **→ JEE Mains 2014**

(a) $40(\sqrt{2}-1)$ (b) $40(\sqrt{3}-\sqrt{2})$ (c) $20\sqrt{2}$ (d) $20(\sqrt{3}-1)$

19 The shadow of a pole of height $(\sqrt{3}+1)$ m standing on the ground is found to be 2 m longer, when the elevation is 30° than when elevation was α , then α is equal to

(a) 15° (b) 30° (c) 45° (d) 75°

20 If the angles of elevation of the top of a tower from three collinear points A, B and C on a line leading to the foot of the tower are 30°,45° and 60° respectively, then the ratio $AB : BC$ is **→ JEE Mains 2015**

(a) $\sqrt{3}:1$ (b) $\sqrt{3}:\sqrt{2}$ (c) $1:\sqrt{3}$ (d) $2:3$

21 From the top of a tower 100 m height, the angles of depression of two objects 200 m apart on the horizontal plane and in a line passing through the foot of the tower and on the same side of the tower are $45°-A$ and $45°+A$. Then, the angle A is equal to

(a) 25° (b) 30° (c) $22\frac{1}{2}°$ (d) 45°

22 Two vertical poles 20 m and 80 m stands apart on a horizontal plane. The height of the point of intersection of the lines joining the top of each pole to the foot of the other is

(a) 15 m (b) 16 m (c) 18 m (d) 50 m

23 Let a vertical tower AB have its end A on the level ground. Let C be the mid-point of AB and P be a point on the ground such that $AP = 2AB$. If $\angle BPC = \beta$, then $\tan\beta$ is equal to **→ JEE Mains 2017**

(a) $\frac{6}{7}$ (b) $\frac{1}{4}$ (c) $\frac{2}{9}$ (d) $\frac{4}{9}$

24 From the tower 60 m high angles of depression of the top and bottom of a house are α and β, respectively. If the height of the house is $\frac{60\sin(\beta-\alpha)}{x}$, then x is equal to

(a) $\sin\alpha\sin\beta$ (b) $\cos\alpha\cos\beta$
(c) $\sin\alpha\cos\beta$ (d) $\cos\alpha\sin\beta$

25 A vertical tower stands on a declivity which is inclined at 15° to the horizon. From the foot of the tower a man ascends the declivity for 80 ft and then, finds that the tower subtends an angle of 30°. The height of tower is

(a) $20(\sqrt{6}-\sqrt{2})$ ft (b) $40(\sqrt{6}-\sqrt{2})$ ft
(c) $40(\sqrt{6}+\sqrt{2})$ ft (d) None of these

26 A ladder rest against a wall at an $\angle\alpha$ to the horizontal. Its foot is pulled away through a distance a_1, so that it slides a distance b_1 down the wall and rests inclined at $\angle\beta$ with the horizontal. It foot is further pulled aways through a_2, so that it slides a further distance b_2 down the wall and is now, inclined at an $\angle\gamma$. If $a_1 a_2 = b_1 b_2$, then

(a) $\alpha+\beta+\gamma$ is greater than π
(b) $\alpha+\beta+\gamma$ is equal to π
(c) $\alpha+\beta+\gamma$ is less than π
(d) nothing can be said about $\alpha+\beta+\gamma$

27 $ABCD$ is a trapezium such that AB and CD are parallel and $BC\perp CD$. If $\angle ADB = \theta$, $BC = p$ and $CD = q$, then AB is equal to **→ JEE Mains 2013**

(a) $\frac{(p^2+q^2)\sin\theta}{p\cos\theta+q\sin\theta}$ (b) $\frac{p^2+q^2\cos\theta}{p\cos\theta+q\sin\theta}$

(c) $\frac{p^2+q^2}{p^2\cos\theta+q^2\sin\theta}$ (d) $\frac{(p^2+q^2)\sin\theta}{(p\cos\theta+q\sin\theta)^2}$

28 The angle of elevation of the top of the tower observed from each of the three points A, B, C on the ground forming a triangle is the same $\angle\alpha$. If R is the circumradius of the ΔABC, then the height of the tower is

(a) $R\sin\alpha$ (b) $R\cos\alpha$
(c) $R\cot\alpha$ (d) $R\tan\alpha$

29 A tower stands at the centre of a circular park. A and B are two points on the boundary of the park such that $AB(=a)$ subtends an angle of 60° at the foot of the tower and the angle of elevation of the top of the tower from A or B is 30°. The height of the tower is

(a) $\frac{2a}{\sqrt{3}}$ (b) $2a\sqrt{3}$ (c) $\frac{a}{\sqrt{3}}$ (d) $\sqrt{3}$

30 $ABCD$ is a square plot. The angle of elevation of the top of a pole standing at D from A or C is 30° and that from B is θ, then $\tan\theta$ is equal to

(a) $\sqrt{6}$ (b) $1/\sqrt{6}$ (c) $\sqrt{3}/2$ (d) $\sqrt{2/3}$

DAY PRACTICE SESSION 2

PROGRESSIVE QUESTIONS EXERCISE

1 For a regular polygon, let r and R be the radii of the inscribed and the circumscribed circles. A false statement among the following is

(a) there is a regular polygon with $\frac{r}{R} = \frac{1}{2}$

(b) there is a regular polygon with $\frac{r}{R} = \frac{1}{\sqrt{2}}$

(c) there is a regular polygon with $\frac{r}{R} = \frac{2}{3}$

(d) there is a regular polygon with $\frac{r}{R} = \frac{\sqrt{3}}{2}$

2 If $\cos A + \cos B + 2\cos C = 2$, then the sides of the ΔABC are in

(a) AP (b) GP
(c) HP (d) None of these

3 If A_0, A_1, A_2, A_3, A_4 and A_5 are the consecutive vertices of a regular hexagon inscribed in a unit circle. Then, the product of length of $A_0A_1 \times A_0A_2$ is

(a) $\sqrt{3}$ (b) $2\sqrt{3}$ (c) 2 (d) $3\sqrt{3}$

4 Two ships leave a port at the same time.One goes 24 km/h in the direction North 45° East and other travels 32 km/h in the direction South 75° East. The distance between the ships at the end of 3 h is approximately

(a) 81.4 km (b) 82 km (c) 85 km (d) 86.4 km

5 A round balloon of radius r subtends an angle α at the observer, while the angle of elevation of its centre is β. The height of the centre of balloon is

(a) $r \operatorname{cosec}\alpha \sin\frac{\beta}{2}$ (b) $r\sin\beta \operatorname{cosec}\frac{\alpha}{2}$

(c) $r\sin\frac{\alpha}{2}\operatorname{cosec}\beta$ (d) $r\operatorname{cosec}\frac{\alpha}{2}\sin\beta$

6 PQR is a triangular park with $PQ = PR = 200$ m. A TV tower stands at the mid-point of QR. If the angles of elevation of the top of the tower at P, Q and R are respectively 45°, 30° and 30°, then the height of the tower (in m) is **→ JEE Mains 2018**

(a) 100 (b) 50
(c) $100\sqrt{3}$ (d) $50\sqrt{2}$

7 At the foot of a mountain the elevation of its summit is 45°. After ascending 2 km towards the mountain up an incline of 30°, the elevation changes to 60°. The height of mountain is

(a) 2.732 km (b) 1.732 km
(c) 2.03 km (d) 1.045 km

8 In a cubical hall $ABCDPQRS$ with each side 10 m, G is the centre of the wall $BCRQ$ and T is the mid-point of the side AB. The angle of elevation of G at the point T is

(a) $\sin^{-1}(1/\sqrt{3})$ (b) $\cos^{-1}(1/\sqrt{3})$
(c) $\cot^{-1}(1/\sqrt{3})$ (d) None of these

9 A tree stands vertically on a hill side which makes an angle of 15° with the horizontal.From a point on the ground 35 m down the hill from the base of the tree, the angle of elevation of the top of the tree is 60°.The height of the tree is

(a) $31\sqrt{2}$ m (b) $33\sqrt{2}$ m
(c) $35\sqrt{2}$ m (d) $34\sqrt{2}$ m

10 A flag staff stands in the centre of a rectangular field whose diagonal is 1200 m, and subtends angles 15° and 45° at the mid-points of the sides of the field. The height of the flag staff is

(a) 200 m (b) $300(\sqrt{2+\sqrt{3}})$ m
(c) $300(\sqrt{2-\sqrt{3}})$ m (d) 400 m

ANSWERS

SESSION 1	**1.** (a)	**2.** (a)	**3.** (a)	**4.** (d)	**5.** (c)	**6.** (c)	**7.** (d)	**8.** (b)	**9.** (a)	**10.** (c)
	11. (c)	**12.** (a)	**13.** (b)	**14.** (a)	**15.** (a)	**16.** (a)	**17.** (d)	**18.** (d)	**19.** (c)	**20.** (a)
	21. (c)	**22.** (b)	**23.** (c)	**24.** (d)	**25.** (b)	**26.** (c)	**27.** (a)	**28.** (d)	**29.** (c)	**30.** (b)

SESSION 2	**1.** (c)	**2.** (a)	**3.** (a)	**4.** (d)	**5.** (b)	**6.** (a)	**7.** (a)	**8.** (a)	**9.** (c)	**10.** (c)

Hints and Explanations

SESSION 1

1 Clearly, $(\sqrt{a}+\sqrt{b}+\sqrt{c})(\sqrt{a}+\sqrt{b}-\sqrt{c})$

$$= (\sqrt{a}+\sqrt{b})^2 - c$$

$$= a + b - c + 2\sqrt{ab} > 0$$

$\therefore \sqrt{a}+\sqrt{b} > \sqrt{c}$

[$\because a+b>c$, as sides of triangle]

2 Given, $A = 30°, B = 45°, a = 1$

$\therefore \quad C = 105°$

Using sine rule,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}, \text{ we get}$$

$$\frac{1}{\left(\frac{1}{2}\right)} = \frac{b}{\left(\frac{1}{\sqrt{2}}\right)} = \frac{c}{\sin 105°}$$

$\Rightarrow \quad b = \sqrt{2}$

and

$c = 2\sin 105° = 2\cos 15°$

$$\therefore \quad c = 2\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right) = \frac{\sqrt{3}+1}{\sqrt{2}}$$

3 Given, $A = 75°$ and $B = 45°$

$\Rightarrow \quad C = 60°$

By sine rule, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$, we get $\frac{a}{\sin 75°} = \frac{b}{\sin 45°} = \frac{c}{\sin 60°}$

$$\text{Now, } b + c\sqrt{2} = \frac{\sin 45°}{\sin 75°}a + \sqrt{2}\,\frac{\sin 60°}{\sin 75°}a$$

$$= \frac{\frac{1}{\sqrt{2}}}{\frac{\sqrt{3}+1}{2\sqrt{2}}}a + \sqrt{2}\,\frac{\frac{\sqrt{3}}{2}}{\frac{\sqrt{3}+1}{2\sqrt{2}}}a$$

$$= \frac{2}{\sqrt{3}+1}a + \frac{2\sqrt{3}a}{\sqrt{3}+1} = 2a$$

4 $$\frac{\sin 3B}{\sin B} = \frac{3\sin B - 4\sin^3 B}{\sin B}$$

$$= 3 - 4\sin^2 B$$

$$= 3 - 4(1-\cos^2 B)$$

$$= -1 + \frac{4(a^2+c^2-b^2)^2}{4(ac)^2}$$

$$= -1 + \frac{\left(\frac{a^2+c^2}{2}\right)^2}{(ac)^2}$$

[$\because 2b^2 = a^2 + c^2$ (given)]

$$= -1 + \frac{(a^2+c^2)^2}{4(ac)^2}$$

$$= \frac{(a^2+c^2)^2 - 4a^2c^2}{4(ac)^2} = \left(\frac{c^2-a^2}{2ac}\right)^2$$

5 Let $a = \sin\alpha, b = \cos\alpha$

and $c = \sqrt{1+\sin\alpha\cos\alpha}$

Here, we can see that the greatest side is c.

$$\therefore \quad \cos C = \frac{a^2+b^2-c^2}{2ab}$$

$$\Rightarrow \cos C = \frac{\sin^2\alpha + \cos^2\alpha - 1 - \sin\alpha\cos\alpha}{2ab}$$

$$\Rightarrow \quad \cos C = -\frac{\sin\alpha\cos\alpha}{2\sin\alpha\cos\alpha}$$

$$\Rightarrow \quad \cos C = -\frac{1}{2} = \cos 120°$$

$\Rightarrow \quad \angle C = 120°$

6 We have,

$$2s(2s-2c) = ab \Rightarrow \frac{s(s-c)}{ab} = \frac{1}{4}$$

$$\Rightarrow \quad \cos^2\frac{C}{2} = \frac{1}{4}$$

$$\text{we get, } \cos\frac{C}{2} = \frac{1}{2}$$

$\left[\text{since, } \frac{C}{2} \text{ must be acute}\right]$

$$\Rightarrow \quad \frac{C}{2} = 60° \Rightarrow C = \frac{2\pi}{3}$$

7 We have, $\tan\frac{A}{2}\tan\frac{C}{2} = \frac{1}{2}$

$$\Rightarrow \quad \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}\sqrt{\frac{(s-a)(s-b)}{s(s-c)}} = \frac{1}{2}$$

$$\Rightarrow \quad \frac{s-b}{s} = \frac{1}{2} \Rightarrow 2s - 2b - s = 0$$

$\Rightarrow \quad a + c - 3b = 0$

8 We have, $R = \frac{abc}{4\Delta}$ and $r = \frac{\Delta}{s}$

$$\therefore \quad \frac{R}{r} = \frac{abc}{4\Delta}\cdot\frac{s}{\Delta} = \frac{abc}{4(s-a)(s-b)(s-c)}$$

Since, $a:b:c = 4:5:6$

$$\Rightarrow \quad \frac{a}{4} = \frac{b}{5} = \frac{c}{6} = k \qquad \text{(say)}$$

$$\text{Thus, } \frac{R}{r} = \frac{(4k)(5k)(6k)}{4\left(\frac{15k}{2}-4k\right)\left(\frac{15k}{2}-5k\right)\left(\frac{15k}{2}-6k\right)}$$

$$= \frac{120k^3\cdot 2}{k^3\cdot 7\cdot 5\cdot 3} = \frac{16}{7}$$

9 We have, $4\left(\frac{s}{a}-1\right)\left(\frac{s}{b}-1\right)\left(\frac{s}{c}-1\right)$

$$= 4\cdot\frac{(s-a)(s-b)(s-c)}{abc}$$

$$= 4\cdot\frac{s(s-a)(s-b)(s-c)}{s\cdot abc}$$

$$= 4\cdot\frac{\Delta^2}{s\cdot abc} = \frac{4\Delta}{abc}\cdot\frac{\Delta}{s} = \frac{r}{R}$$

10 We have, $\frac{a\cos A + b\cos B + c\cos C}{a+b+c}$

$$= \frac{2R\sin A\cos A + 2R\sin B\cos B + 2R\sin C\cos C}{2s}$$

$$= \frac{R}{2s}\cdot(\sin 2A + \sin 2B + \sin 2C)$$

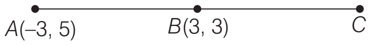
$$= \frac{R}{2s}\cdot 4\sin A\sin B\sin C$$

$$= \frac{2R}{s}\cdot\frac{abc}{8R^3} = \frac{abc}{4sR^2}$$

$$= \frac{4\Delta R}{4\cdot\frac{\Delta}{r}\cdot R^2} = \frac{r}{R} \quad \left[\because R = \frac{abc}{4\Delta}, r = \frac{\Delta}{s}\right]$$

11 **Key Idea** Orthocentre, centroid and circumcentre are collinear and centroid divide orthocentre and circumcentre in 2:1 ratio.

We have orthocentre $A(-3,5)$ and centroid $B(3,3)$. Let C be the circumcentre.



Clearly, $AB = \sqrt{(3+3)^2 + (3-5)^2}$

$= \sqrt{36+4} = 2\sqrt{10}$

We know that, $AB:BC = 2:1 \Rightarrow BC = \sqrt{10}$

Now, $AC = AB + BC = 2\sqrt{10} + \sqrt{10}$

$= 3\sqrt{10}$

Since, AC is a diameter of circle.

$\therefore \quad r = \frac{AC}{2} \Rightarrow r = \frac{3\sqrt{10}}{2} = 3\sqrt{\frac{5}{2}}$

12 Since, $\left(1 - \frac{r_1}{r_2}\right)\left(1 - \frac{r_1}{r_3}\right) = 2$

$\therefore \quad \left(1 - \frac{s-b}{s-a}\right)\left(1 - \frac{s-c}{s-a}\right) = 2$

$\Rightarrow \quad \frac{(b-a)(c-a)}{(s-a)^2} = 2$

$\Rightarrow \quad \frac{bc - ab - ac + a^2}{\left(\frac{a+b+c}{2} - a\right)^2} = 2$

$\Rightarrow \quad \frac{2(bc - ab - ac + a^2)}{(b+c-a)^2} = 1$

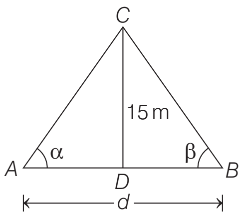
$\Rightarrow \quad 2bc - 2ab - 2ac + 2a^2$

$= b^2 + c^2 + a^2 + 2bc - 2ab - 2ac$

$\Rightarrow a^2 = b^2 + c^2$

So, triangle is right angled.

13 Given, $\cot\alpha = \frac{12}{5}$ and $\sin\beta = \frac{3}{5}$



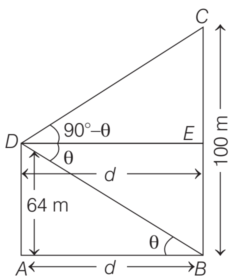
In ΔDAC and ΔDBC,

$AD = 15\cot\alpha,\ BD = 15\cot\beta$

$\Rightarrow \quad d = 15(\cot\alpha + \cot\beta)$

$= 15\left(\frac{12}{5} + \frac{4}{3}\right) = 56 \text{ m}$

14 In ΔDAB, $\tan\theta = \frac{64}{d}$



$\Rightarrow \quad d = 64\cot\theta \quad \ldots\text{(i)}$

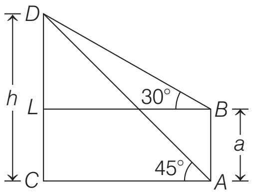
In ΔCDE, $\tan(90° - \theta) = \frac{(100-64)}{d}$

$\Rightarrow \quad d = 36\tan\theta \quad \ldots\text{(ii)}$

From Eqs. (i) and (ii), we get

$d^2 = 36 \times 64 \Rightarrow d = 48 \text{ m}$

15 Let CD be a tower of height h and AB is building of height a.



In ΔBLD, $\tan 30° = \frac{h-a}{LB}$

$\therefore \quad LB = \frac{(h-a)}{\tan 30°} = \sqrt{3}(h-a) \quad \ldots\text{(i)}$

In ΔACD, $\tan 45° = \frac{h}{CA} \Rightarrow h = CA \Rightarrow$

$h = LB \quad [\because LB = CA] \ldots\text{(ii)}$

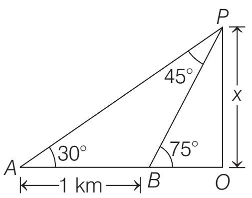
From Eqs. (i) and (ii), we get

$h(\sqrt{3} - 1) = \sqrt{3}\,a$

$\therefore \quad h = \frac{\sqrt{3}\,a}{\sqrt{3} - 1} = \frac{\sqrt{3}(\sqrt{3} + 1)\,a}{2}$

$= \left(\frac{3 + \sqrt{3}}{2}\right) a$

16 Let OP be the pole of height x m.



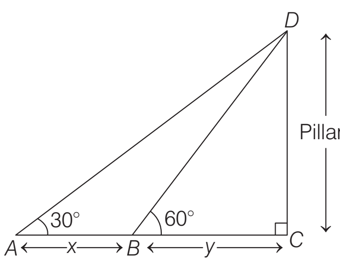
Using sine rule in APB, $\frac{\sin 30°}{\sin 45°} = \frac{PB}{AB}$

$\Rightarrow \quad PB = 1000 \times \frac{1/2}{1/\sqrt{2}} = 500\sqrt{2} \text{ m}$

In ΔPBO, $x = 500\sqrt{2} \cdot \sin 75°$

$= 500\sqrt{2} \times \frac{\sqrt{3}+1}{2\sqrt{2}} = 250(\sqrt{3}+1) \text{ m}$

17 According to given information, we have the following figure



Now, from ΔACD and ΔBCD, we have

$\tan 30° = \frac{h}{x+y}$

and $\quad \tan 60° = \frac{h}{y}$

$\Rightarrow \quad h = \frac{x+y}{\sqrt{3}} \quad \ldots\text{(i)}$

and $\quad h = \sqrt{3}y \quad \ldots\text{(ii)}$

From Eqs. (i) and (ii)

$\frac{x+y}{\sqrt{3}} = \sqrt{3}y$

$\Rightarrow \quad x + y = 3y$

$\Rightarrow \quad x - 2y = 0 \Rightarrow y = \frac{x}{2}$

$\because$ Speed is uniform

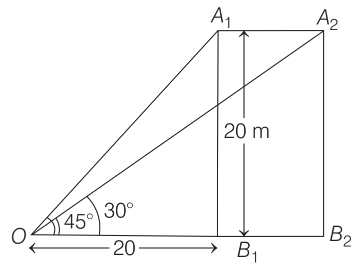
$\therefore$ Distance y will be cover in 5 min.

($\because$ Distance x covered in 10 min.)

$\Rightarrow$ Distance $\frac{x}{2}$ will be cover in 5 min.

18 In ΔOA_1B_1, $\tan 45° = \frac{A_1B_1}{OB_1}$

$\Rightarrow \quad \frac{20}{OB_1} = 1 \Rightarrow OB_1 = 20$



In ΔOA_2B_2,

$\tan 30° = \frac{20}{OB_2} \Rightarrow OB_2 = 20\sqrt{3}$

$\Rightarrow \quad B_1B_2 + OB_1 = 20\sqrt{3}$

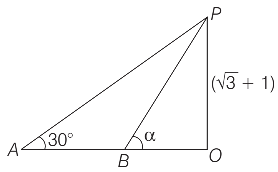
$\Rightarrow \quad B_1B_2 = 20\sqrt{3} - 20$

$\Rightarrow \quad B_1B_2 = 20(\sqrt{3} - 1)\text{m}$

Now, Speed $= \frac{\text{Distance}}{\text{Time}} = \frac{20(\sqrt{3}-1)}{1}$

$= 20(\sqrt{3} - 1) \text{ m/s}$

19 Let OP be a tower with height $(\sqrt{3} + 1)$ m and $AB = 2$ m.



In ΔAOP,

$\tan 30° = \frac{\sqrt{3}+1}{OA}$

$\Rightarrow \quad OA = (\sqrt{3} + 1)\sqrt{3}$

and in ΔBOP, $\tan\alpha = \dfrac{\sqrt{3}+1}{OB}$

$\Rightarrow \quad OB = (\sqrt{3}+1)\cot\alpha$

Now,

$OA - OB = (3+\sqrt{3}) - (\sqrt{3}+1)\cot\alpha$

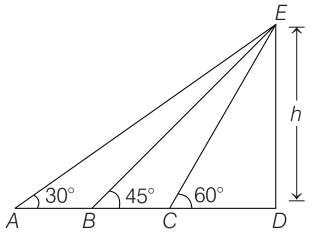
$\Rightarrow \quad 2 = 3 + \sqrt{3} - (\sqrt{3}+1)\cot\alpha$

$\Rightarrow \quad \cot\alpha = 1$

$\therefore \quad \alpha = 45^\circ$

20 According to the given information the figure should be as follows.

Let the height of tower $= h$



In ΔEDA, $\tan 30^\circ = \dfrac{ED}{AD}$

$\Rightarrow \quad \dfrac{1}{\sqrt{3}} = \dfrac{ED}{AD} = \dfrac{h}{AD} \Rightarrow AD = h\sqrt{3}$

In ΔEDB, $\tan 45^\circ = \dfrac{h}{BD} \Rightarrow BD = h$

In ΔEDC, $\tan 60^\circ = \dfrac{h}{CD} \Rightarrow CD = \dfrac{h}{\sqrt{3}}$

Now, $\dfrac{AB}{BC} = \dfrac{AD - BD}{BD - CD} \Rightarrow \dfrac{AB}{BC} = \dfrac{h\sqrt{3} - h}{h - \dfrac{h}{\sqrt{3}}}$

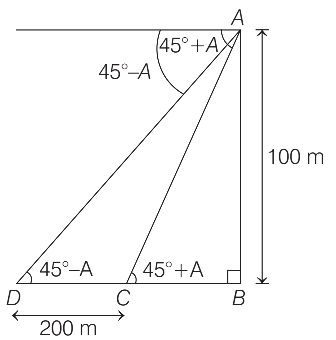
$\Rightarrow \quad \dfrac{AB}{BC} = \dfrac{h(\sqrt{3}-1)}{\dfrac{h(\sqrt{3}-1)}{\sqrt{3}}} \Rightarrow \dfrac{AB}{BC} = \dfrac{\sqrt{3}}{1}$

$\therefore \quad AB : BC = \sqrt{3} : 1$

21 Let AB be the tower with height 100 m.

Distance between the objects,

$CD = 200$ m



In ΔACB, $\tan(45^\circ + A) = \dfrac{100}{BC}$

$\Rightarrow \quad BC = 100\cot(45^\circ + A) \quad \ldots(\text{i})$

and in ΔADB,

$\tan(45^\circ - A) = \dfrac{100}{BD}$

$\Rightarrow \quad BD = 100\cot(45^\circ - A)$

$\Rightarrow \quad BC + CD = 100\cot(45^\circ - A) \quad \ldots(\text{ii})$

$\therefore$ From Eqs. (i) and (ii), we get

$CD = 100\,[\cot(45^\circ - A) - \cot(45^\circ + A)]$

$= 100\left[\dfrac{1+\tan A}{1-\tan A} - \dfrac{1-\tan A}{1+\tan A}\right]$

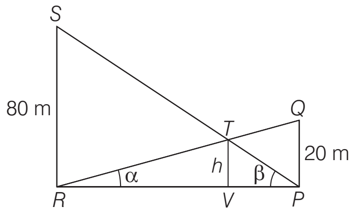
$= 100\left[\dfrac{4\tan A}{1-\tan^2 A}\right] = 200\tan 2A$

$\Rightarrow \quad 200 = 200\tan 2A \Rightarrow \tan 2A = 1$

$\Rightarrow \quad \tan 2A = \tan 45^\circ \Rightarrow 2A = 45^\circ$

$\therefore \quad A = 22\dfrac{1}{2}^\circ$

22 Let PQ and RS be the poles of height 20 m and 80 m subtending angles α and β at R and P, respectively. Let h be the height of the point T, the intersection of QR and PS.



Then, $PR = h\cot\alpha + h\cot\beta$

$= 20\cot\alpha = 80\cot\beta$

$\Rightarrow \quad \cot\alpha = 4\cot\beta \Rightarrow \dfrac{\cot\alpha}{\cot\beta} = 4$

Again, $h\cot\alpha + h\cot\beta = 20\cot\alpha$

$\Rightarrow \quad (h-20)\cot\alpha = -h\cot\beta$

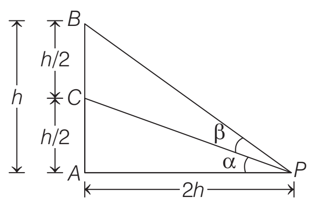
$\Rightarrow \quad \dfrac{\cot\alpha}{\cot\beta} = \dfrac{h}{20-h} = 4$

$\Rightarrow \quad h = 80 - 4h$

$\therefore \quad h = 16$ m

23 Let $AB = h$, then $AP = 2h$ and $AC = BC = \dfrac{h}{2}$

Again, let $\angle CPA = \alpha$



Now, in ΔABP,

$\tan(\alpha+\beta) = \dfrac{AB}{AP} = \dfrac{h}{2h} = \dfrac{1}{2}$

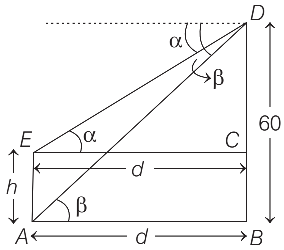
Also, in ΔACP, $\tan\alpha = \dfrac{AC}{AP} = \dfrac{\frac{h}{2}}{2h} = \dfrac{1}{4}$

Now, $\tan\beta = \tan[(\alpha+\beta) - \alpha]$

$= \dfrac{\tan(\alpha+\beta) - \tan\alpha}{1 + \tan(\alpha+\beta)\tan\alpha} = \dfrac{\frac{1}{2} - \frac{1}{4}}{1 + \frac{1}{2}\times\frac{1}{4}} = \dfrac{\frac{1}{4}}{\frac{9}{8}}$

$= \dfrac{2}{9}$

24 In ΔABD, $\tan\beta = \dfrac{60}{d}$



$\Rightarrow \quad d = 60\cot\beta \quad \ldots(\text{i})$

In ΔDEC, $\tan\alpha = \dfrac{DC}{EC}$

$\Rightarrow \quad DC = d\tan\alpha$

$\Rightarrow \quad 60 - h = d\tan\alpha \quad (\because BC = EA = h)$

$\Rightarrow \quad 60 - h = 60\cot\beta\tan\alpha \quad$ [from Eq. (i)]

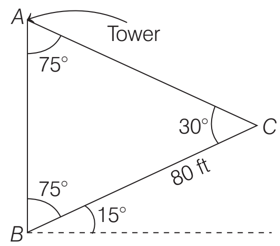
$\Rightarrow \quad h = 60\left(1 - \dfrac{\cos\beta}{\sin\beta}\cdot\dfrac{\sin\alpha}{\cos\alpha}\right)$

$\Rightarrow \quad h = \dfrac{60\sin(\beta-\alpha)}{\cos\alpha\sin\beta}$

$\Rightarrow \quad \dfrac{60\sin(\beta-\alpha)}{x} = \dfrac{60\sin(\beta-\alpha)}{\cos\alpha\sin\beta}$ (given)

$\Rightarrow \quad x = \cos\alpha\sin\beta$

25 Let BC be the declivity and BA be the tower.



In ΔABC, on applying sine rule

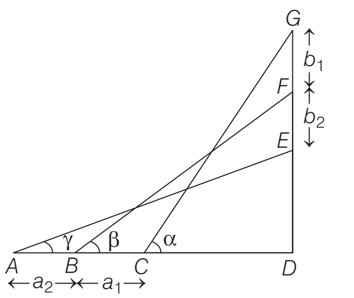
$\dfrac{BC}{\sin 75^\circ} = \dfrac{AB}{\sin 30^\circ}$

$\Rightarrow \quad AB = \dfrac{80\sin 30^\circ}{\sin 75^\circ} = \dfrac{40\times 2\sqrt{2}}{\sqrt{3}+1}$

$= 40(\sqrt{6} - \sqrt{2})$ ft

26 Clearly, $\dfrac{a_1}{b_1} = \tan\left(\dfrac{\alpha+\beta}{2}\right) \quad \ldots(\text{i})$

and $\quad \dfrac{a_2}{b_2} = \tan\left(\dfrac{\beta+\gamma}{2}\right) \quad \ldots(\text{ii})$



Since, $a_1 a_2 = b_1 b_2$

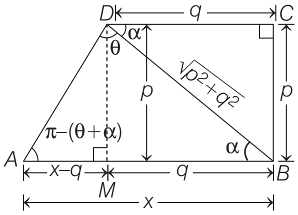
$$\therefore \frac{a_1}{b_1} = \frac{b_2}{a_2} \Rightarrow \tan\left(\frac{\alpha+\beta}{2}\right) = \frac{1}{\tan\left(\frac{\beta+\gamma}{2}\right)}$$

$$\Rightarrow \quad \tan\left(\frac{\alpha+\beta}{2}\right)\tan\left(\frac{\beta+\gamma}{2}\right) = 1$$

$$\therefore \quad \frac{\alpha+\beta}{2} + \frac{\beta+\gamma}{2} = \frac{\pi}{2}$$

$$\Rightarrow \quad \alpha+\beta+\gamma = \pi - \beta < \pi$$

27 Let $AB = x$



In ΔDAM, $\tan(\pi - \theta - \alpha) = \frac{p}{x-q}$

$$\Rightarrow \quad \tan(\theta+\alpha) = \frac{p}{q-x}$$

$$\Rightarrow \quad q - x = p\cot(\theta+\alpha)$$

$$\Rightarrow \quad x = q - p\cot(\theta+\alpha)$$

$$= q - p\left(\frac{\cot\theta\cot\alpha - 1}{\cot\alpha + \cot\theta}\right)$$

$$= q - p\left(\frac{\frac{q}{p}\cot\theta - 1}{\frac{q}{p} + \cot\theta}\right)$$

$$\left[\because \text{in } \Delta BDC, \cot\alpha = \frac{q}{p}\right]$$

$$= q - p\left(\frac{q\cot\theta - p}{q + p\cot\theta}\right).$$

$$= q - p\left(\frac{q\cos\theta - p\sin\theta}{q\sin\theta + p\cos\theta}\right)$$

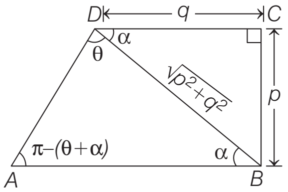
$$\Rightarrow \quad x = \frac{q^2\sin\theta + pq\cos\theta - pq\cos\theta + p^2\sin\theta}{p\cos\theta + q\sin\theta}$$

$$\Rightarrow \quad AB = \frac{(p^2+q^2)\sin\theta}{p\cos\theta + q\sin\theta}$$

Alternate Solution

Applying sine rule in ΔABD,

$$\left(\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}\right)$$



$$\frac{AB}{\sin\theta} = \frac{\sqrt{p^2+q^2}}{\sin\{\pi - (\theta+\alpha)\}}$$

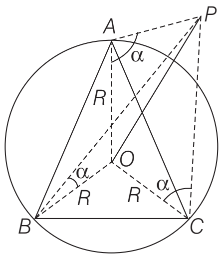
$$\Rightarrow \quad \frac{AB}{\sin\theta} = \frac{\sqrt{p^2+q^2}}{\sin(\theta+\alpha)}$$

$$\Rightarrow \quad AB = \frac{\sqrt{p^2+q^2}\sin\theta}{\sin\theta\cos\alpha + \cos\theta\sin\alpha}$$

$$= \frac{(p^2+q^2)\sin\theta}{q\sin\theta + p\cos\theta}$$

$$\left[\because \cos\alpha = \frac{q}{\sqrt{p^2+q^2}}, \sin\alpha = \frac{p}{\sqrt{p^2+q^2}}\right]$$

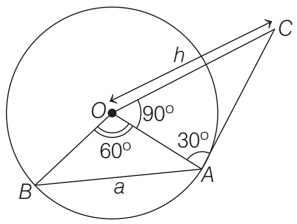
28 Let OP be the tower. Since, the tower make equal angles at the vertices of the triangle, therefore foot of tower is at the circumcentre.



In ΔOAP, $\tan\alpha = \frac{OP}{OA} \Rightarrow OP = OA\tan\alpha$

$$\Rightarrow \quad OP = R\tan\alpha \quad (\because OA = R, \text{given})$$

29 Let h be the height of a tower



Since, $\angle AOB = 60°$

Also, $OB = OA =$ radii

$\therefore \quad \angle OBA = \angle OAB = 60°$

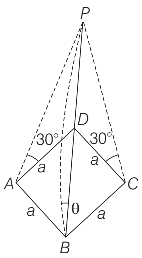
So, ΔOAB is an equilateral

$\therefore \quad OA = OB = AB = a$

In ΔOAC, $\tan 30° = \frac{h}{a}$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{h}{a} \Rightarrow h = \frac{a}{\sqrt{3}}$$

30 Let PD be a pole.



In ΔDAP, $\tan 30° = \frac{DP}{AD}$

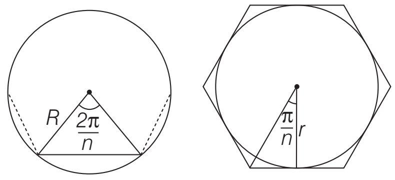
$$\Rightarrow \quad DP = \frac{a}{\sqrt{3}}$$

In ΔPDB, $\tan\theta = \frac{DP}{BD}$

$$\Rightarrow \quad \tan\theta = \frac{a/\sqrt{3}}{\sqrt{2}a} = \frac{1}{\sqrt{6}}$$

SESSION 2

1



$$\therefore \quad \frac{a}{2R} = \sin\frac{\pi}{n} \text{ and } \frac{a}{2r} = \tan\frac{\pi}{n}$$

$$\therefore \quad \frac{r}{R} = \cos\frac{\pi}{n}$$

$n = 3$ gives, $\frac{r}{R} = \frac{1}{2}$

$n = 4$ gives, $\frac{r}{R} = \frac{1}{\sqrt{2}}$

$n = 6$ gives, $\frac{r}{R} = \frac{\sqrt{3}}{2}$

2 $2\cos\frac{A+B}{2}\cos\frac{A-B}{2} = 4\sin^2\frac{C}{2}$

$$\Rightarrow \quad \cos\frac{A-B}{2} = 2\cos\frac{A+B}{2}$$

$$\left[\because \cos\frac{A+B}{2} \neq 0\right]$$

$$\Rightarrow \quad \cos\frac{A}{2}\cos\frac{B}{2} + \sin\frac{A}{2}\cdot\sin\frac{B}{2} = 2\left(\cos\frac{A}{2}\cos\frac{B}{2} - \sin\frac{A}{2}\cdot\sin\frac{B}{2}\right)$$

$$\Rightarrow \quad 3\sin\frac{A}{2}\cdot\sin\frac{B}{2} = \cos\frac{A}{2}\cos\frac{B}{2}$$

$$\Rightarrow \quad \tan\frac{A}{2}\cdot\tan\frac{B}{2} = \frac{1}{3}$$

$$\Rightarrow \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}\cdot\sqrt{\frac{(s-c)(s-a)}{s(s-b)}} = \frac{1}{3}$$

$$\Rightarrow \quad \frac{s-c}{s} = \frac{1}{3} \Rightarrow \frac{a+b-c}{a+b+c} = \frac{1}{3}$$

$$\Rightarrow \quad 3a + 3b - 3c = a + b + c$$

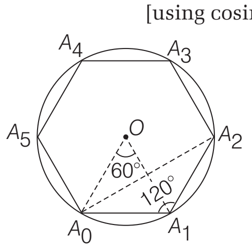
$$\Rightarrow \quad a + b = 2c$$

Hence, a, c and b are in AP.

3 Clearly,

$A_0A_1 = 2 \times 1\cos 60° = 1 = A_1A_2$

[using cosine rule]]



and

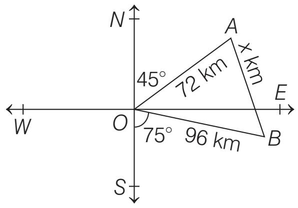
$$\cos 120^\circ = \frac{(A_0A_1)^2 + (A_1A_2)^2 - (A_0A_2)^2}{2 \cdot A_0A_1 \cdot A_1A_2}$$

$$= \frac{1 + 1 - (A_0A_2)^2}{2 \cdot 1 \cdot 1}$$

$\Rightarrow\ A_0A_2 = \sqrt{3}$

$\therefore\ A_0A_1 \times A_0A_2 = 1 \times \sqrt{3} = \sqrt{3}$

4 Let A and B be the positions of the two ships at the end of 3 h.
Then, $OA = (24 \times 3) = 72$ km and $OB = (32 \times 3) = 96$ km



Let $AB = x$ km

We have, $\angle NOA = 45^\circ$ and $\angle SOB = 75^\circ$

$\therefore\ \angle AOB = 180^\circ - (45^\circ + 75^\circ) = 60^\circ$

Using cosine formula on ΔAOB, we get

$$\cos 60^\circ = \frac{(72)^2 + (96)^2 - x^2}{2 \times 72 \times 96}$$

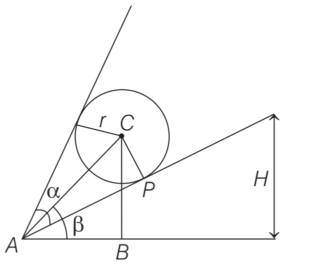
$\Rightarrow\ x^2 = 14400 - 6912 = 7488$

$\Rightarrow\ x = \sqrt{7488} = 86.53$ km

Hence, the distance between the ships at the end of 3 h is
86.53 km ≈ 86.4 km (approx.).

5 In ΔAPC,

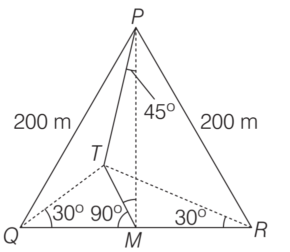
$$\sin \frac{\alpha}{2} = \frac{r}{AC} \Rightarrow AC = r \operatorname{cosec} \frac{\alpha}{2}$$



In $\Delta\ ABC$, $\sin \beta = \dfrac{BC}{AC}$

$\Rightarrow\ H = AC \sin \beta \Rightarrow H = r \sin \beta \operatorname{cosec} \dfrac{\alpha}{2}$

6



Let height of tower TM be h.

In ΔPMT, $\tan 45^\circ = \dfrac{TM}{PM}$

$\Rightarrow\ 1 = \dfrac{h}{PM}$

$\Rightarrow\ PM = h$

In ΔTQM, $\tan 30^\circ = \dfrac{h}{QM}$; $QM = \sqrt{3}h$

In ΔPMQ, $PM^2 + QM^2 = PQ^2$

$h^2 + (\sqrt{3}h)^2 = (200)^2$

$\Rightarrow\ 4h^2 = (200)^2 \Rightarrow h = 100$ m

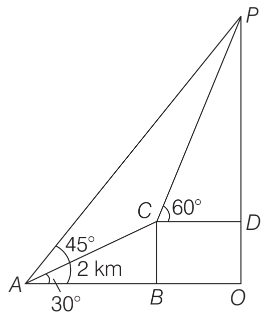
7 Let P be the summit and A be the foot of the mountain.

Then, $\angle OAP = 45^\circ$

$\therefore\ \angle OPA = 45^\circ$

Hence, ΔOAP is an isosceles.

Let $AO = OP = h$ km height of mountain. [say]



Let C be the point where elevation is 60°.

Then, $\angle BAC = 30^\circ$ and $AC = 2$ km

$\therefore\ \dfrac{BC}{AC} = \sin 30^\circ$

$\Rightarrow\ BC = 2 \times \dfrac{1}{2} = 1$

i.e. $BC = 1$ km and $\dfrac{AB}{AC} = \cos 30^\circ$

$\Rightarrow\ AB = \dfrac{2\sqrt{3}}{2} = \sqrt{3}$

i.e. $AB = \sqrt{3}$ km

Now, $PD = OP - OD$

$= OP - BC = h - 1$

and $CD = BO = AO - AB = h - \sqrt{3}$

In ΔDCP, $\dfrac{PD}{CD} = \tan 60^\circ \Rightarrow \dfrac{h-1}{h-\sqrt{3}} = \sqrt{3}$

$\Rightarrow\ \sqrt{3}h - 3 = h - 1 \Rightarrow (\sqrt{3} - 1)h = 2$

$\therefore\ h = \dfrac{2}{\sqrt{3}-1} \times \dfrac{\sqrt{3}+1}{\sqrt{3}+1} = \sqrt{3} + 1$

$= 1.732 + 1 = 2.732$ km

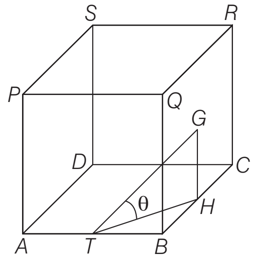
8 Let H be the mid-point of BC. Since, $\angle TBH = 90^\circ$, therefore,

$TH^2 = BT^2 + BH^2 = 5^2 + 5^2 = 50$

Also since,

$\angle THG = 90^\circ$, $TG^2 = TH^2 + GH^2$

$= 50 + 25 = 75$

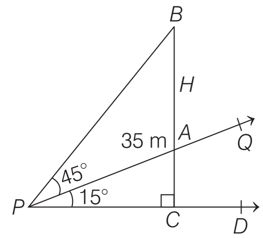


Let θ be the required angle of elevation of G at T.

Then, $\sin\theta = \dfrac{GH}{TG} = \dfrac{5}{5\sqrt{3}} = \dfrac{1}{\sqrt{3}}$

$\Rightarrow\ \theta = \sin^{-1}(1/\sqrt{3})$

9 Let PAQ be the hill, AB be the tree and PD be the horizontal. Let P be the point of observation.



Produce BA to meet PD at C.

Let $AB = H$ m.

Then, $\angle DPA = 15^\circ$, $PA = 35$m

$\angle CPB = 60^\circ$ and $\angle PCA = 90^\circ$.

$\therefore\ \angle APB = (60^\circ - 15^\circ) = 45^\circ$

In ΔPAC, $\angle PAC = 180^\circ - (15^\circ + 90^\circ) = 75^\circ$

$\therefore\ \angle PAB = (180^\circ - 75^\circ) = 105^\circ$

and $\angle PBA = 180^\circ - (45^\circ + 105^\circ) = 30^\circ$

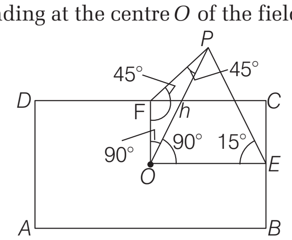
Applying sine rule on ΔPAB, we get

$$\frac{PA}{\sin \angle PBA} = \frac{AB}{\sin \angle APB} \Rightarrow \frac{35}{\sin 30^\circ} = \frac{H}{\sin 45^\circ}$$

$\therefore\ 35 \times 2 = H \times \sqrt{2} \Rightarrow H = 35\sqrt{2}$ m

Hence, the height of the tree is $35\sqrt{2}$ m.

10 Let OP be the flag staff of height h standing at the centre O of the field.



In $\Delta\ OEP$, $OE = h \cot 15^\circ = h(2 + \sqrt{3})$

and in $\Delta\ OFP$, $OF = h \cot 45^\circ = h$

$\therefore\ EF = h\sqrt{1 + (2+\sqrt{3})^2} = 2h\sqrt{2+\sqrt{3}}$

Since, $BD = 1200$ m

$\Rightarrow\ 2EF = 4h\sqrt{(2+\sqrt{3})} = 1200$

$\therefore\ h = \dfrac{300}{\sqrt{2+\sqrt{3}}} = (300\sqrt{2-\sqrt{3}})$ m